WRITTEN HOMEWORK #6, DUE FEB 17, 2010

(1) (Chapter 17.2, #45)
(a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
(b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y\rangle$?

(2) (Chapter 17.3, #14) Let $\mathbf{F}(x, y) = \langle \frac{y^2}{1+x^2}, 2y \arctan x\rangle$, and let C be parameterized by $\mathbf{r}(t) = \langle t^2, 2t\rangle, 0 \le t \le 1$.
(a) Find a function f such that $\mathbf{F} = \nabla f$, and
(b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along C.

(3) For each of the following five regions D, draw a sketch of each region, and identify whether they are connected and/or simply-connected.
(a) $\{(x, y)|y \neq 0\}$; ie, $\mathbb{R}^2$ with the x axis removed.
(b) $1 \le x^2 + y^2 \le 4, y \ge 0$.
(c) $x \ge 4$ or $x < 3$.
(d) The region D given by polar inequalities $r \ge 0, \pi/4 \le \theta \le 3\pi/4$.
(e) $\{(x, y)|(x, y) \neq (x, 0), x < 0\}$; ie, $\mathbb{R}^2$ with the negative x axis removed.

(4) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\rangle.$$

This vector field appears in a Webwork homework assignment, and in that assignment you showed it was not conservative on $\mathbb{R}^2$ with the origin removed. Let D be the region $\{(x, y)|(x, y) \neq (x, 0), x < 0\}$; ie, $\mathbb{R}^2$ with the negative x axis removed (the same region as part (e) of the previous problem).
(a) Explain why $\mathbf{F}$ is conservative on D.
(b) For (x, y) in D, let $x = r \cos \theta, y = r \sin \theta$, where r, θ are polar coordinates, with θ restricted by $-\pi < \theta < \pi$. (Notice that $\theta = \pi$ corresponds to the negative x-axis, which is not in D.) Suppose $f(x, y)$ is a potential function for $\mathbf{F}$, which satisfies $f(1, 0) = 0$. Show that $f(x, y) = \theta$ by calculating the line integral of a path from $(1, 0)$ to (x, y). (Hint: pick a path which goes from $(1, 0)$ to $(r, 0)$, and then goes to (x, y) by traveling along the circle of radius r.)

(5) (Chapter 17.4, #12) Let $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x\rangle$, and let C be the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$. (That is, the orientation of C is given by this traversal of the listed vertices.) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(6) (Chapter 17.4, #21) (a) If C is the line segment connecting the starting point (x_1, y_1) to end point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1.$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots + (x_{n-1}yn - x_ny_{n-1}) + (x_ny_1 - x_1y_n)).$$

(c) Find the area of the quadrilateral with vertices $(0,0),(2,1),(1,3),(-1,1)$. (The textbook gives a pentagon with one additional vertex, but the pentagon really is a quadrilateral because three of the listed vertices are collinear.)